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Higher-order fluxes and the speed of thermal waves

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Abstract—By following the formalism of extended irreversible thermodynamics, we obtain a hierarchy of evolution equations for the higher-order fluxes in heat conduction. We study the influence of the higher-order fluxes on the speed of thermal waves in the second-, third- and fourth-order approximations, as well as in the asymptotic limit of infinite higher-order fluxes.

INTRODUCTION

HEAT WAVES and hyperbolic heat transport are very active topics of research, from the experimental, theoretical and computational points of view. The literature on the subject has been recently reviewed in refs. [1, 2], and other useful references covering a wide literature are refs. [3–6]. The need of transport equations leading to a finite signal speed in heat conduction has been one of the incentives to develop a new thermodynamic formalism, known as extended irreversible thermodynamics (EIT) [1, 7–12], which goes beyond the local-equilibrium assumption which is the starting point of the classical irreversible thermodynamics [13].

In the standard version of EIT [1], the usual dissipative fluxes (heat flux, diffusion flux, viscous pressure tensor, etc.) are considered as independent thermodynamic variables. A generalized nonequilibrium entropy is defined which depends not only on its classical variables, but also on the dissipative fluxes. Whereas the evolution of the former ones is described by the balance laws of mass, momentum and energy, the equations describing the evolution of the latter ones must be found for each material in such a way that they are compatible with the second law of thermodynamics. The inclusion of the heat flux $\mathbf{q}$ as independent variable leads, in the simplest situation, to an evolution equation for $\mathbf{q}$ which is precisely the well-known Maxwell–Cattaneo equation

$$\tau_1 \, d\mathbf{q}/dt + \mathbf{q} = -\lambda \nabla T \tag{1}$$

with τ_1 a relaxation time, λ the thermal conductivity and T the absolute temperature. When equation (1) is introduced into the energy balance equation

$$\rho c \, dT/dt = -\nabla \cdot \mathbf{q} \tag{2}$$

with ρ the mass density and c a specific heat per unit mass, it leads to a finite speed for thermal signals (the high-frequency limit of the phase speed of thermal waves), given by

$$v_{MC}^2 = (\chi/\tau_1) \tag{3}$$

with $\chi = \lambda/(\rho c)$ the thermal diffusivity.

One of the problems in equation (1) is to determine the relaxation time τ_1. In the simplest versions, it may be identified with the mean-free time between consecutive collisions, or, in more elaborate versions, with the inverse of the eigenvalue of the collision operator associated with $\mathbf{q}$.

The problem we deal with here is the following one. The relaxation time τ_1 is certainly of the order of the mean-free time between collision, τ_{col}. Since the relaxation times of the higher-order fluxes (the flux of the heat flux and so on) will also be of the order of τ_{col}, it turns out that in the situations when the frequency is of the order of $1/\tau_{col}$, not only the heat flux $\mathbf{q}$, but all the higher-order fluxes should be considered as independent variables. We have formulated [14–18] the corresponding version of EIT and we have explored some of its consequences up to the second order, i.e. up to the inclusion of the flux of the heat flux and the heat flux itself as independent variables.

The purpose of this paper is to study the influence of the higher-order fluxes on the speed of thermal signals, both in the lower-order approximations as in the many-order, asymptotic limit.

In the first section we write the fundamental hypotheses of EIT and the corresponding evolution equations for the fluxes. In the second, we obtain the dispersion equations for thermal waves in the second-, third- and fourth-order approximations. In the third section we write the frequency- and wavelength-dependent thermal conductivity, and use an asymptotic development for continued-fraction expansions to obtain an effective time for the relaxation of the heat flux taking into account the effect of an infinite number of higher-order fluxes.

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NOMENCLATURE

c	specific heat per unit mass
D_i	elementary hyperbolic operator
$H_n(\infty, k)$	nth-order continued fraction expansion of the thermal conductivity with unit numerator
$\mathbf{J}^s$	entropy flux
k	wave vector
k_B	Boltzmann constant
$\mathbf{q}$	heat flux
$\mathbf{q}_n$	nth-order flux
$\delta\mathbf{q}_n$	fluctuation of $\mathbf{q}$ around equilibrium
s	specific entropy per unit mass
$\delta^2 s$	second differential of the entropy
T	absolute temperature
u	specific internal energy per unit mass
v	specific volume per unit mass
v_{MC}	speed of thermal pulses in Maxwell–Cattaneo theory.

Greek symbols

α_n	entropy coefficients
β_n	entropy flux coefficients
θ_n	phenomenological coefficients in the evolution equations
λ	thermal conductivity
λ_n	higher-order transport coefficients in the evolution equations
$\lambda(\omega, u)$	frequency- and wave vector-dependent conductivity
ρ	mass density
σ	entropy production per unit volume and time
τ_1	relaxation time of the heat flux
τ_{ef}	effective relaxation time of the heat flux
τ_n	relaxation time of the nth order flux
τ_{col}	mean-free time
χ	thermal diffusivity
ω	frequency.

Other symbols

$\Box_i$	D'Alembertian operator
∂_t, ∂_x	partial derivative with respect to time, space.

EXTENDED THERMODYNAMICS WITH HIGHER-ORDER FLUXES

In contrast with the usual nonequilibrium thermodynamics, based on the local-equilibrium hypothesis, EIT [1] assumes that, out of equilibrium, the entropy is no longer the local-equilibrium one but that it depends on the dissipative fluxes. In some situations, these fluxes may be related to some internal variables of the system: the viscous pressure tensor in polymeric solutions, for instance, is related to the conformational tensor of the macromolecules. In other situations, as in monatomic ideal gases, the fluxes do not bear relation to any internal variable at all.

To keep a maximum simplicity in the arguments, we will restrict ourselves to heat transport in a rigid heat conductor. The usual version of extended irreversible thermodynamics takes as independent variables the specific internal energy per unit mass, u, and the heat flux $\mathbf{q}$, and assumes a nonequilibrium specific entropy s which depends on u and $\mathbf{q}$, $s(u, \mathbf{q})$. In a more general version [15], one takes as independent variables u, $\mathbf{q}_1$, $\mathbf{q}_2$, ..., $\mathbf{q}_n$, ... with $\mathbf{q}_n$ (an nth order tensor) the flux of $\mathbf{q}_{n-1}$ (an $n-1$ order tensor). Here, $\mathbf{q}_1$ is identified with $\mathbf{q}$. Note that in a system of N particles, a macroscopic description is based on a small number of variables (mass, energy, momentum, etc.), whereas from a microscopic point of view one would need $6N$ variables. The inclusion of more and more higher-order fluxes provides different mesoscopic descriptions intermediate between the macroscopic (thermodynamic) one and the microscopic one.

In accordance with the postulates of EIT [1], one assumes a generalized Gibbs entropy of the form [15]

$$ds = T^{-1}\,du - v\sum_{n=1}^{\infty}\alpha_n\mathbf{q}_n\cdot d\mathbf{q}_n \tag{4}$$

with T absolute temperature, v specific volume and α_n a set of coefficients, which may depend in principle on u, and on the scalar invariants of the tensors, and whose physical meaning will be seen in the evolution equations for the fluxes. A dot between two tensors means their total mutual contraction to give a scalar, i.e. $\mathbf{q}_n\cdot\mathbf{q}_n = q_{i_1i_2\ldots i_N}q_{i_1i_2\ldots i_N}$, with summation over repeated indices.

As well as the entropy, the entropy flux $\mathbf{J}^s$ is assumed to depend on all dissipative fluxes. Thus, one assumes that

$$\mathbf{J}^s = T^{-1}\mathbf{q}_1 - \sum_{1}^{\infty}\beta_n\mathbf{q}_{n+1}\cdot\mathbf{q}_n \tag{5}$$

where the dot means the total contraction of $\mathbf{q}_{n+1}$ and $\mathbf{q}_n$ to give a vector, i.e. $\mathbf{q}_{n+1}\cdot\mathbf{q}_n = q_{i_1\ldots i_{N+1}}q_{i_1\ldots i_N}$ with summation over repeated indices. The β_n are parameters (possibly functions of u and of the scalar invariants of the fluxes), whose meaning will also be identified in the evolution equations of the fluxes.

When the energy balance equation (2) with $c\,dT = du$ is used, one finds for the time derivative of s

$$\rho\,ds/dt = -\nabla\cdot(T^{-1}\mathbf{q}) + \mathbf{q}\cdot\nabla T^{-1} - \sum_{n=1}^{\infty}\alpha_n\mathbf{q}_n\cdot d\mathbf{q}_n/dt. \tag{6}$$

The entropy production σ is found, as usual, from the general form of a balance equation

$$\rho\, ds/dt + \nabla\cdot \mathbf{J}^s = \sigma. \quad (7)$$

Introduction of equations (5) and (6) into equation (7) yields

$$\sigma = -\mathbf{q}_1\cdot[T^{-2}\nabla T + \alpha_1\, d\mathbf{q}_1/dt + \beta_1\nabla\cdot\mathbf{q}_2] - \sum_{n=2}^{\infty}\mathbf{q}_n\cdot[\alpha_n\, d\mathbf{q}_n/dt + \beta_n\nabla\cdot\mathbf{q}_{n+1} + \nabla(\beta_{n-1}\mathbf{q}_{n-1})]. \quad (8)$$

Note that up to the Nth order, one assumes that $\mathbf{q}_M = 0$ for $M \geqslant N+1$, and the term $\mathbf{q}_N\beta_N\nabla\cdot\mathbf{q}_{N+1}$ would not be present in equation (8). In equation (8) we assume tacitly that this absence is not important in the limit when $N\to\infty$. This assumption is common to all higher-order kinetic theory developments [19], but it is in fact difficult to show rigorously.

We will assume here a linear situation, in which the parameters involved in the theory do not depend on the fluxes themselves. Inclusion of some nonlinear effects leads to features not included in the linear theory as, for instance [16] a difference in the speed of thermal waves along and against a superimposed temperature gradient. The aim of this paper is to point the influence of higher-order fluxes on the speed of high-frequency, low-amplitude thermal waves, so that the complications of a nonlinear theory may be avoided here.

The simplest set of linear evolution equations compatible with the positive definite character of the entropy production (8) is

$$T^{-2}\nabla T + \alpha_1\, d\mathbf{q}_1/dt + \beta_1\nabla\cdot\mathbf{q}_2 = -\theta_1\mathbf{q}_1 \quad (9)$$

$$\alpha_n\, d\mathbf{q}_n/dt + \beta_n\nabla\cdot\mathbf{q}_{n+1} + \nabla(\beta_{n-1}\mathbf{q}_{n-1}) = -\theta_n\mathbf{q}_n \quad (10)$$

with $\theta_n \geqslant 0$. Note that θ_1 may be identified as $(\lambda T^2)^{-1}$ and the relaxation times τ_n of $\mathbf{q}_n$ as $\tau_n = (\alpha_n/\theta_n)$. The definition of $\mathbf{q}_{n+1}$ as the flux of $\mathbf{q}_n$ implies $\beta_n = \theta_n$. Then, one may rewrite equations (9) and (10) as

$$\tau_1\, d\mathbf{q}_1/dt = -\lambda\nabla T - \mathbf{q}_1 - \nabla\cdot\mathbf{q}_2 \quad (11)$$

$$\tau_n\, d\mathbf{q}_n/dt = -\lambda_n\nabla\mathbf{q}_{n-1} - \mathbf{q}_n - \nabla\cdot\mathbf{q}_{n-1} \quad (12)$$

with $\lambda_n = (\theta_{n-1}/\theta_n) = (\tau_n/\tau_{n-1})(\alpha_{n-1}/\alpha_n)$. Note that an alternative definition of $\mathbf{q}_n$ as flux could have been to assume that $\beta_n = \alpha_n$; the corresponding set of equations would be essentially equivalent to the present one. This hierarchy of equations, where the time derivative of $\mathbf{q}_n$ depends on $\mathbf{q}_{n+1}$, will be the basis of our discussion. Note that if one assumes $\mathbf{q}_i = 0$ for $i \geqslant 2$, equations (11) and (12) reduce to the Maxwell–Cattaneo expression (1).

We have achieved a description of the evolution of the system in terms of two sets of parameters: λ_n and τ_n (or alternatively α_n and τ_n). The parameters α_n may be easily related to the second moments of the fluctuations of the fluxes around the equilibrium. Indeed, by writing the Einstein formula for the probability of fluctuations [1]

$$Pr \sim \exp[\delta^2 s/2k_B] \quad (13)$$

with k_B the Boltzmann constant, one finds

$$Pr(\delta\mathbf{q}_i) \sim \exp[-(v\alpha_n/2k_B)\delta\mathbf{q}_n\cdot\delta\mathbf{q}_n]. \quad (14)$$

The second moments of the fluctuations may be easily related to the α_n as

$$\langle\delta\mathbf{q}_{nxx\ldots x}\delta\mathbf{q}_{nxx\ldots x}\rangle = (2k/v\alpha_n) \quad (15)$$

with the angular brackets standing for an equilibrium average and $\delta\mathbf{q}_n$ the fluctuation of the $xx\ldots x$ components of $\mathbf{q}_n$. Relation (15) allows the evaluation from purely equilibrium statistical mechanics the parameters α_n, whereas the relaxation times τ_n are of dynamical nature and are related to the collision operator characteristic of the system.

LOWER-ORDER APPROXIMATIONS

To see in an explicit way how the presence of higher-order fluxes influences the speed of thermal waves, we will first write equations (11) and (12) in a unidimensional case up to the fourth order. We have

$$\begin{aligned}
\rho c\partial_t T &= 0 - \partial_x\mathbf{q}_1 + 0 + 0\\
\tau_1\partial_t\mathbf{q}_1 &= -\lambda\partial_x T - \mathbf{q}_1 - \partial_x\mathbf{q}_2 + 0\\
\tau_2\partial_t\mathbf{q}_2 &= 0 - \lambda_2\partial_x\mathbf{q}_1 - \mathbf{q}_2 - \partial_x\mathbf{q}_3\\
\tau_3\partial_t\mathbf{q}_3 &= 0 + 0 - \lambda_3\partial_x\mathbf{q}_2 - \mathbf{q}_3
\end{aligned} \quad (16)$$

with $\lambda_2 = (\tau_2/\tau_1)(\alpha_1/\alpha_2)$ and $\lambda_3 = (\tau_3/\tau_2)(\alpha_2/\alpha_3)$. We will write explicitly the second-, third- and fourth-order approximations, and will compare them with the general form [20]

$$\begin{aligned}
M^{(1)}T &= \eta_1 D_1 T + \eta_0 T\\
M^{(2)}T &= \eta_2 D_2 D_3 T + \eta_1 D_1 T + \eta_0 T\\
M^{(3)}T &= \eta_3 D_4 D_5 D_6 T + \eta_2 D_2 D_3 T + \eta_1 D_1 T + \eta_0 T\\
M^{(4)}T &= \eta_4 D_7 D_8 D_9 D_{10} T + \eta_3 D_4 D_5 D_6 T\\
&\quad + \eta_2 D_2 D_3 T + \eta_1 D_1 T + \eta_0 T = 0
\end{aligned} \quad (17)$$

with $D_i = \partial_t + c_i\partial_x$. The symbols ∂_t and ∂_x denote respectively the partial derivatives with respect to t and x.

The second-order approximation ($\mathbf{q}_2 = \mathbf{q}_3 = 0$) to equation (16) is

$$\Box_2 + \eta_1\partial_t = M_J^{(2)} \quad (18)$$

with $\Box_i = \partial_t^2 - c_i^2\partial_x^2$ the D'Alembert operator related to the speed c_i. Here, $\eta_1 = \tau_1^{-1}$ and the speed of the waves is

$$c_2^2 = (\chi/\tau_1) \quad (19)$$

which is the speed (2) of the classical Maxwell–Cattaneo equation.

Up to the third-order approximation ($\mathbf{q}_3 = 0$), one has

$$\partial_t \Box_4 + \eta'_2 \Box_2 + \eta'_1 \partial_t = M_J^{(3)} \quad (20)$$

where

$$\eta'_2 = \tau_1^{-1} + \tau_2^{-1}, \eta'_1 = (\tau_2\tau_1)^{-1}$$
$$c_2'^2 = \chi[\tau_1 + \tau_2]^{-1}$$

and the signal speed

$$c_4'^2 = (\chi/\tau_1) + (\lambda_2/\tau_1\tau_2) \quad (21)$$

instead of equation (19).

In the fourth-order approximation ($\mathbf{q}_n = 0$), i.e. the full (16) expression, one has

$$\Box_7 \Box_9 + \eta''_3 \partial_t \Box_4 + \eta''_2 \Box_2 + \eta''_1 \partial_t = M_J^{(4)} \quad (22)$$

with

$$\eta''_3 = \tau_1^{-1} + \tau_2^{-1} + \tau_3^{-1}, \eta''_2 = (\tau_1\tau_2)^{-1} + (\tau_2\tau_3)^{-1} + (\tau_1\tau_3)^{-1}, \eta''_1 = (\tau_1\tau_2\tau_3)^{-1}$$
$$c_2''^2 = \chi[\tau_1 + \tau_2 + \tau_3]^{-1},$$
$$c_4''^2 = [(\tau_2 + \tau_3)\chi + (\lambda_2 + \lambda_3)][\tau_1 + \tau_2 + \tau_3]^{-1}$$

and the signal speeds c''_7 and c''_9 given by

$$c_7''^2 + c_9''^2 = (\chi/\tau_1) + (\lambda_2/\tau_1\tau_2) + (\lambda_3/\tau_2\tau_3)$$
$$c_7''^2 c_9''^2 = (\chi/\tau_1)(\lambda_3/\tau_2\tau_3). \quad (23)$$

We see that the system is not strictly hyperbolic, in the sense that the velocities of the characteristics are degenerate, but symmetric hyperbolic. In ref. [20], an expression for the Lagrangians leading to these equations is given.

Expressions (19), (21) and (23) show the influence of the first higher-order fluxes on the speed of thermal waves. As an explicit illustration, we may consider the equations used to describe phonon hydrodynamics in dielectric solids [1, 16, 21]. Such equations yield, in the notation used in this paper, to

$$\chi = (1/3)c_0^2\tau_R, \quad \lambda_2 = (1/5)c_0^2\tau_N\tau_R$$

with c_0 the phonon velocity, and τ_R and τ_N the resistive and normal collision times, respectively, which turn out to be the relaxation times of $\mathbf{q}_1$ and of $\mathbf{q}_2$, respectively. Thus, the speed in the first-order approximation, as given by equation (19), is $c_1^2 = (1/3)c_0^2$ whereas in the second-order approximation one has from equation (21) $c_2^2 = (8/15)c_0^2$. To our knowledge, only these two fluxes have been considered in dealing with solids.

How many higher-order fluxes should be included in the description depends on the spectrum of the collision operator. If there are a few fluxes with relatively long times and the other ones have much shorter relaxation times, it is sufficient to keep only the former ones as independent variables. However, in the kinetic theory of gases, all the higher-order fluxes have relaxation times of the same order, at least in the relaxation-time approximation, and even in more complicated models [19]. Thus, to consider only a few independent higher-order fluxes is, in principle, conceptually inconsistent: an infinite number of them should be taken into account. Thus, an asymptotic development for the speed of thermal waves should be devised.

ASYMPTOTIC EXPRESSION AND EFFECTIVE RELAXATION TIME

The hierarchy (16) may be written as a generalized frequency- and wave vector-dependent thermal conductivity $\lambda(\omega, k)$. Such generalized coefficients are much used in generalized hydrodynamics and in rheology. We may write the hierarchy (11) and (12) in the Fourier space, assuming vanishing initial values of the $\mathbf{q}_n$, as

$$i\omega\tau_1\hat{\mathbf{q}}_1 = -i\mathbf{k}\lambda\hat{T} - \hat{\mathbf{q}}_1 - i\mathbf{k}\cdot\hat{\mathbf{q}}_2$$
$$i\omega\tau_n\hat{\mathbf{q}}_n = i\mathbf{k}\lambda_n\hat{\mathbf{q}}_{n-1} - \hat{\mathbf{q}}_n - i\mathbf{k}\cdot\hat{\mathbf{q}}_{n+1} \quad (24)$$

with ω the frequency, k the wave vector and $\hat{}$ the corresponding Fourier transform. From equation (24) one may write for the Fourier transform of the heat flux

$$\hat{\mathbf{q}}_1 = -i\mathbf{k}\lambda(\omega, k)\hat{T} \quad (25)$$

with $\lambda(\omega, k)$ given by

$$\lambda(\omega, k) = \cfrac{\lambda}{1 + i\omega\tau_1 + \cfrac{l_1^2k^2}{1 + i\omega\tau_2 + \cfrac{l_2^2k^2}{1 + i\omega\tau_3 + \cfrac{l_3^2k^2}{1 + i\omega\tau_4 + \ldots}}}} \quad (26)$$

with $l_n^2 = (\tau_n\alpha_{n-1}/\tau_{n-1}\alpha_n) = \lambda_n$. The truncations of this continued fraction should be made by assuming that all τ_i up to order n and all l_i^2 up to order $n-1$ are different from zero, and that $l_n^2 = 0$. The simplest non-trivial approximations to equation (26) are

$$\lambda_1(\omega, k) = \frac{\lambda}{1 + i\omega\tau_1} \quad (27)$$

corresponding to the Maxwell–Cattaneo approximation. The second approximation would be

$$\lambda_2(\omega, k) = \cfrac{\lambda}{1 + i\omega\tau_1 + \cfrac{l_1^2k^2}{1 + i\omega\tau_2}}. \quad (28)$$

One may obtain an asymptotic expression for equation (26) by using the scheme proposed in ref. [22]. We define $\lambda_n(\omega, k)$ as the nth order approximation to equation (26) ($\tau_m = 0$ for $m \geqslant n+1$, $l_m^2 = 0$ for $m \geqslant n$) and we define $H_n(\omega, k)$ as

$$H_n(\omega, k) = (1/\lambda)\lambda_n(\omega, k) \quad (29)$$

so that $H_n(\omega, k)$ is a continued fraction of order n with the first numerator normalized to unity. In the asymptotic limit, one assumes [22]

$$H_\infty(\omega,k) = \frac{1}{1+i\omega\tau_\infty + l_\infty^2 k^2 H_\infty(\omega,k)}. \quad (30)$$

This scheme has proved sufficiently accurate in several areas of physics [22].

From equation (30) we have

$$H_\infty(\omega,k) = \frac{-(1+i\omega\tau_\infty) \pm [(1+i\omega\tau_\infty)^2 + 4l_\infty^2 k^2]^{1/2}}{2l_\infty^2 k^2} \quad (31)$$

with τ_∞ and l_∞^2 the values of τ_n and l_n^2 in the limit of high n.

The dispersion equation for thermal waves in this limit is

$$\rho c i\omega = -\lambda k^2 H_\infty(\omega,k). \quad (32)$$

In the high-frequency limit ($\omega\tau_\infty \gg 1$), equations (31) and (32) yield

$$i\omega = (\chi/2l_\infty^2)[i\omega\tau_\infty \pm (4l_\infty^2 k^2 - \omega^2\tau_\infty^2)^{1/2}]. \quad (33)$$

This leads for the phase speed to

$$c_\infty^2 = [\chi^2/l_\infty^2][(\chi\tau_\infty/l_\infty^2)-1]^{-1}. \quad (34)$$

This expression allows one to define an effective or 'renormalized' time which takes into account the effect of all the higher-order fluxes on the relaxation of $\mathbf{q}$. We define τ_{ef}, the effective time, as

$$c_\infty^2 = (\chi/\tau_{ef}) \quad (35)$$

in accordance with equation (3). Comparison of equations (34) and (35) yields

$$\tau_{ef} = \tau_\infty - (l_\infty^2/\chi). \quad (36)$$

In the case when all the relaxation times have the same value (relaxation-time approximation) and if l_∞^2 is very small, τ_{ef} is equal to the relaxation time or collision time. Some effective relaxation times have been computed in very different situations in ref. [23].

CONCLUSIONS

The development presented here is of interest, because it allows in a certain way to reduce the set of infinite number of fast variables to a few fast variables (in our case, to the heat flux). Undoubtedly, much information is lost in this reduction procedure, but at least the speed signals include the most relevant complexities of the system. The situation is similar to the one presented near critical points: in these ones, the order parameter and all its combinations become slow variables, or, from another point of view, the fluctuations of all scales become important. The renormalization group technique allows one to redefine the parameters of the theory in such a way that reducing the number of variables one does not lose fundamental information. In our case, not only the heat flux but an infinite set of higher-order fluxes become slow variables (in comparison with the frequency of the experiment), and we have tried to obtain a recipe to reduce the description of the system to a small number of variables without losing essential information about the signal speed.

The situation is not always that depicted in this asymptotic situation. In other cases, as in solids at low temperatures [21], there is a big difference between the relaxation time of $\mathbf{q}_1$ and that of $\mathbf{q}_2$ and of the higher-order fluxes. Thus, in this case a low approximation may be valid. In fact, a development with $\mathbf{q}_1$ and $\mathbf{q}_2$ different from zero but with $\tau_2 = 0$ was used by Guyer and Krumhansl to describe second sound and phonon hydrodynamics in dielectric solids at low temperatures [21]. Their equation is a particular case of equation (16), but it is parabolic instead of hyperbolic. In fact, their second sound speed is a plateau limit for frequencies higher than τ_1^{-1} but lower than τ_2^{-1}.

A rough estimation of the effects of equation (36) could be obtained by assuming that $l_\infty^2 = v_x^2\tau^2$ with $v_x^2 = (k_B T/m)$, with m the mass of the molecules and k_B the Boltzmann constant (we assume $l_\infty^2 = v_x^2 l^2$ instead of $l_\infty^2 = v^2\tau^2$ because l_∞ is in fact a projection of a length on the direction of propagation of the wave. In this case one has

$$\tau_{ef} = \tau - (kT/m)(\chi)^{-1}\tau^2.$$

If ones takes into account that in kinetic theory $\chi = (5/3)(kT/m)\tau$, the effective time for heat relaxation as compared with the collision time is

$$\tau_{ef} = \tau - (3/5)\tau = (2/5)\tau.$$

This yields a higher speed for thermal uses than that obtained by the use of τ. More detailed evaluation could be made from, for instance, the Grad moments expansion in kinetic theory [19]. This work is in progress.

An important consequence of the higher speed for thermal waves arising from the consideration of all higher-order fluxes would be found in the context of shock waves in gases. Heuristically, it is known that regular signals cannot propagate in a hyperbolic medium with speed higher than the highest characteristic speed [19]. In the 13-moments development in kinetic theory this leads to the consequence that shock waves cannot have a regular structure for Mach numbers higher than a critical value given by 1.65, in contrast with experimental evidence, which seems to indicate regular structure for higher Mach numbers. Some work in progress by our group shows that by taking into account the development proposed in this paper, the critical Mach number can be raised to 2.86. At such high Mach numbers, the thickness of the shock is only two or three times the mean-free path, and continuum theories are no longer expected to be valid.

Finally, it is worth mentioning, in connection with the development (26) for the generalized thermal conductivity, that continued-fraction expansions of generalized transport coefficients are frequently used in statistical mechanics. Mori [24] was able to show from rather general arguments, by means of projection operator techniques, that continued fraction expansions have a microscopic bias.

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REFERENCES

1. D. Jou, J. Casas-Vazquez and G. Lebon, Extended irreversible thermodynamics, *Rep. Prog. Phys.* **51,** 1105–1179 (1988).
2. D. D. Joseph and L. Preziosi, Heat waves, *Rev. Mod. Phys.* **61,** 41–73 (1989).
3. D. E. Glass, M. N. Ozisik and B. Vick, Non-Fourier effects on transient temperature resulting from periodic on–off heat flux, *Int. J. Heat Mass Transfer* **30,** 1623–1631 (1987).
4. D. Y. Tzou, Thermal shock waves induced by a moving crack—a heat flux formulation, *Int. J. Heat Mass Transfer* **33,** 887–896 (1990).
5. J. Gembarovic and V. Majernik, Non-Fourier propagation of heat pulses in finite medium, *Int. J. Heat Mass Transfer* **31,** 1073–1080 (1988).
6. S. Sieniutycz, The inertial relaxation terms and the variational principle of least action type for nonstationary energy and mass diffusion, *Int. J. Heat Mass Transfer* **26,** 55–63 (1983).
7. I. Muller, *Thermodynamics*. Pitman, London (1985).
8. I. Gyarmati, On the wave approach to thermodynamics and some problems in non-linear theories, *J. Non-Equilib. Thermodyn.* **2,** 233–245 (1977).
9. L. S. Garcia-Colin, Extended irreversible thermodynamics: scope and limitations, *Rev. Mex. Fis.* **34,** 344 (1989).
10. J. Casas-Vazquez, D. Jou and G. Lebon (Editors), *Recent Developments in Nonequilibrium Thermodynamics*. Springer, Berlin (1984).
11. R. E. Nettleton, Relaxation theory of thermal conduction, *Physics Fluids* **3,** 216 (1960).
12. I. Muller and T. Ruggeri (Editors), *Kinetic Theory and Extended Thermodynamics*. Pitagora Editrice, Bologna (1987).
13. S. R. De Groot and P. Mazur, *Nonequilibrium Thermodynamics*. North-Holland, Amsterdam (1962).
14. D. Jou and C. Perez-Garcia, On a Ginzburg–Landau equation for the evolution and the fluctuations of heat flux, *Physica A* **104,** 320–332 (1980).
15. C. Perez-Garcia and D. Jou, Thermodynamic aspects of continued-fraction expansion in heat conduction, *J. Phys. A* **19,** 2881–2890 (1986).
16. D. Jou and J. Casas-Vazquez, Nonequilibrium absolute temperature, thermal waves and phonon hydrodynamics, *Physica A* **163,** 47–58 (1990).
17. D. Jou, J. E. Llebot and J. Casas-Vazquez, Irreversible thermodynamic approach to nonequilibrium heat fluctuations, *Phys. Rev. A* **25,** 508–518 (1982).
18. J. E. Llebot, D. Jou and J. Casas-Vazquez, A thermodynamic approach to heat and electric conduction in solids, *Physica A* **121,** 552–562 (1983).
19. H. Grad, Kinetic theory of gases. In *Handbuch der Physik* (Edited by S. Flugge), Vol. XII. Springer, Berlin (1958).
20. M. Kranys, Causal theories of evolution and wave propagation in mathematical physics, *Appl. Mech. Rev.* **42,** 305–322 (1989).
21. R. A. Guyer and H. Krumhansl, Dispersion relation for second sound in solids, *Phys. Rev. A* **133,** 1411–1417 (1964).
22. P. Gianozzi, C. Grosso, S. Moroni and G. Pastore-Parravicini, The ordinary and matrix continued fractions in the thermodynamic analysis of Hermitian and relaxation operators, *Appl. Numer. Math.* **4,** 273–295 (1988).
23. Z. Banach and S. Pierarski, Irreducible tensor description, *J. Math. Phys.* **30,** 1826–1834 (1989).
24. H. Mori, A continued-fraction representation of the time-correlation functions, *Prog. Theor. Phys.* **33,** 432 (1965).

FLUX D'ORDRE SUPERIEUR ET VITESSE DES ONDES THERMIQUES

Résumé—En accord avec le formalisme de la thermodynamique irreversible étendue (extended irreversible thermodynamics), on obtient une hiérarchie d'équations d'évolution pour les flux thermiques d'ordre supérieur. On étudie l'influence des flux d'ordre supérieur sur la vitesse des ondes thermiques au second, troisième et quatrième ordre, et à la limite asymptotique d'un nombre infini de flux d'ordre supérieur.

STROMDICHTEN HÖHERER ORDNUNG UND DIE GESCHWINDIGKEIT VON TEMPERATURWELLEN

Zusammenfassung—Durch Anwendung der Regeln der erweiterten irreversiblen Thermodynamik wird eine Hierarchie von Entwicklungsgleichungen für Stromdichten höherer Ordnung bei der Wärmeleitung entwickelt. Der Einfluß der Stromdichten höherer Ordnung auf die Ausbreitungsgeschwindigkeit von Temperaturwellen wird anhand von Nährungslösungen zweiter, dritter und vierter Ordnung untersucht, ebenso das asymptotische Verhalten von Stromdichten unendlichgroßer Ordnung.

ПОТОКИ ВЫСОКОГО ПОРЯДКА И СКОРОСТЬ ТЕПЛОВЫХ ВОЛН

Аннотация—На основе формализма теории термодинамики необратимых процессов получена иерархия эволюционных уравнений для потоков высокого порядка в случае теплопроводности. Исследуется влияние этих потоков на скорость тепловых волн в приближениях второго, третьего и четвертого порядка, а также в асимптотическом пределе потоков бесконечно высокого порядка.